Harmonic functions

Tuesday, September 19, 2023 1:19 PM

We will prove later: every analytic function in $B(z_0, r)$ is infinitely differentiable.

A corollary of this:

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\frac{\partial v}{\partial y} = \frac{\partial^2 v}{\partial x \partial y}$$

$$\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x}$$

By Cauchy-Riemann

$\Delta u := \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ – Laplace operator.

Same way:

$\Delta v \equiv 0$

$\Delta f = \Delta u + i \Delta v = 0.$

Examples.

$u = x^2 - y^2$

$u = xy$

$u = e^x \cos y$

Def $u \in C^2$ is called harmonic on a set K if $\Delta u \equiv 0$.

Theorem. Let u be real and harmonic in some $B(z_0, r)$. Then $\exists f$ – analytic in $B(z_0, r)$, $u = \operatorname{Re} f$.

We will prove a more general version later.